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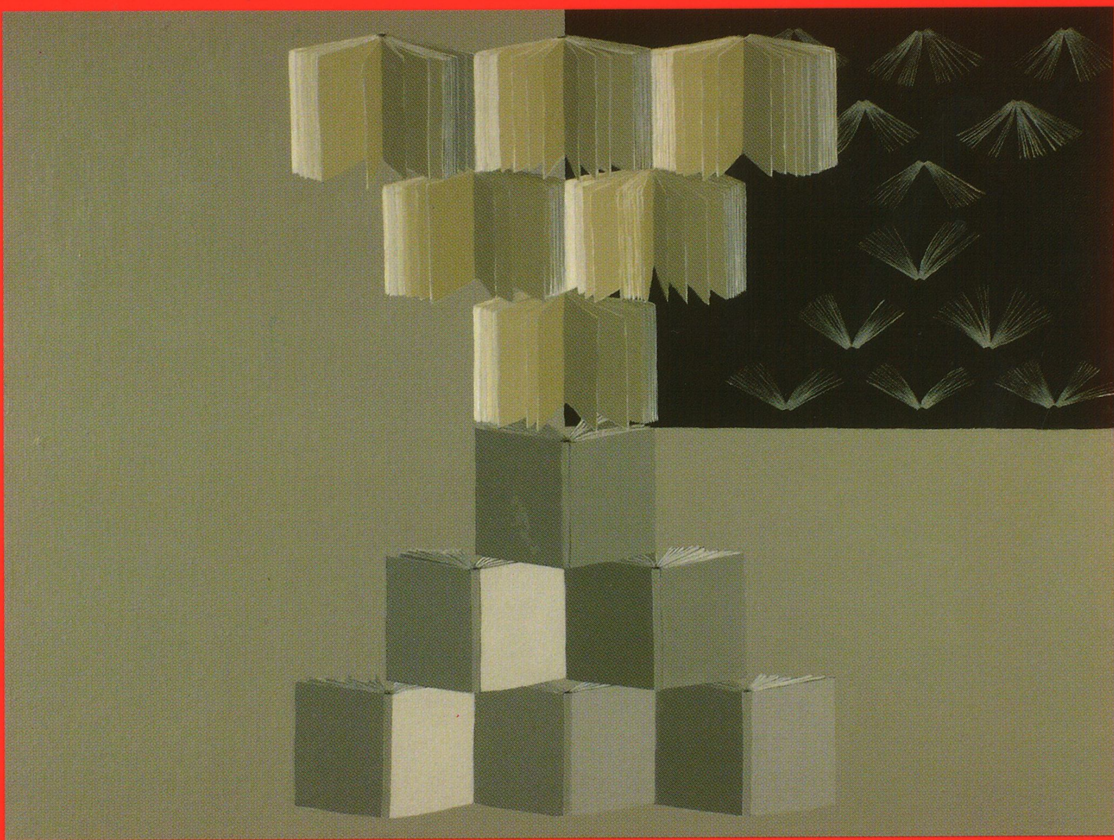
Dipartimento di Matematica



SEMINARI DI GEOMETRIA 2005-2009

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Accessible points, harmonic measure, and the Riemann mapping

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Abstract. Let $n \geq 2$. Let D be a bounded domain in $\mathbb{R}^n$. We provide a bird's-eye view of the relation between harmonic measure in D, the nature of the set of boundary accessible points of D, and, if $n = 2$ and D is simply connected, the Riemann mapping of D. We prove new results and give new, easier proofs of known results. We prove in various ways that the set of boundary accessible points of D (which is not necessarily Borel set if $n > 2$) is indeed measurable for harmonic measure. We also establish precisely for which sets the pullback of harmonic measure under the Riemann mapping is equal to the Lebesgue measure.
Sommario. Sia $n \geq 2$. Sia D un dominio limitato in $\mathbb{R}^n$. Presentiamo una veduta d'insieme della relazione tra la misura armonica relativa a D, la natura dell'insieme dei punti del bordo di D che sono *accessibili*, e, quando $n = 2$ e D è semplicemente connesso, l'applicazione di Riemann di D. Dimostriamo nuovi risultati e offriamo nuove, più semplici dimostrazioni di risultati noti.

This paper is in definitive form and no version of it will be published elsewhere.
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1 Introduction

Let $n \geq 2$ be an integer, and let D be a bounded, connected, and open subset of $\mathbb{R}^n$. In this paper we provide a bird's-eye view of the relation between

1. harmonic measure in D;
2. the nature (in the sense of descriptive set theory) of the set of boundary points which are accessible from D;
3. the Riemann mapping of D (when $n = 2$ and D is simply connected).

Moreover, we prove new results and give new, easier proofs of known theorems.

2 Main Results

The geometric boundary of D is denoted by ∂D. Harmonic measure *for* D (with a given pole) is a particular measure *defined on* ∂D. One of our concerns is given by the following question:

For which subsets of ∂D is harmonic measure well defined?

We adopt the following notation:
$\mathcal{B}(\partial D)$ is the σ-algebra of Borel subsets of ∂D;
$\mathrm{C}(\partial D)$ is the Banach space of real-valued continuous functions on ∂D;
$\mathrm{C}^*(\partial D)$ is the dual of $\mathrm{C}(\partial D)$;
$\mathrm{C}^*_+(\partial D)$ is the cone of positive linear functionals on $\mathrm{C}(\partial D)$;
$\mathrm{h}(D)$ is the vector space of real-valued functions defined and harmonic on D;
$\mathbb{U}$ is the open disc in $\mathbb{R}^2$ of radius one, centered at the origin.
Recall that there exists a unique function[1] $\omega : D \to \mathrm{C}^*(\partial D)$ such that (here we write ω_z for $\omega(z)$)

- for each $z \in D$, ω_z actually belongs to $\mathrm{C}^*_+(\partial D)$; this means that ω_z is a positive measure defined on the Borel σ-algebra $\mathcal{B}(\partial D)$ of ∂D;
- for each $f \in \mathrm{C}(\partial D)$, $\int_{\partial D} f(w)\, \omega_z(dw)$ is a harmonic function of $z \in D$;
- for "most" points $w \in \partial D$, $\lim_{D \ni z \to w} \omega_z = \delta_w$, in the weak-star topology, where $\delta_w \in \mathrm{C}^*(\partial D)$ is the Dirac measure on ∂D concentrated at w.

[1]See Appendix A for a more precise definition.

The positive Borel measure ω_z on ∂D is called the *harmonic measure for* D with *pole* at z. If we need to emphasize the dependence of ω_z on the domain D, we write

$$\omega_z(\mathsf{A} \mid D) \tag{1}$$

instead of $\omega_z(\mathsf{A})$, where $\mathsf{A} \subset \partial D$ is Borel.

As a matter of fact, (1) turns out to be naturally defined not only when A belongs to $\mathcal{B}(\partial D)$, the Borel σ-algebra of ∂D, but *also when* A *belongs to the* σ*-algebra generated by* $\mathcal{B}(\partial D)$ *together with the subsets of Borel subsets of* ∂D *of zero harmonic measure.* Let us denote by $\mathcal{R}_{\partial D}$ this σ-algebra.[2] Then A belongs to $\mathcal{R}_{\partial D}$ if and only if it can be written as

$$\mathsf{A} = \mathsf{A_b} \cup \mathsf{A_o} \tag{2}$$

where $\mathsf{A_b} \in \mathcal{B}(\partial D)$ and $\mathsf{A_o} \subset \partial D$ is a *null set with respect to harmonic measure*, i.e., there exists a set $\mathsf{N} \in \mathcal{B}(\partial D)$ whose harmonic measure equals zero and such that $\mathsf{N} \supset \mathsf{A_o}$. Now, the position

$$\omega_z(\mathsf{A} \mid D) \equiv \omega_z(\mathsf{A_b} \mid D) \tag{3}$$

is a well-defined measure on $\mathcal{R}_{\partial D}$, independent of the particular decomposition used in (2). The Borel σ-algebra $\mathcal{B}(\partial D)$ is strictly contained in $\mathcal{R}_{\partial D}$, and the position in (3) determines as extension of ω_z from $\mathcal{B}(\partial D)$ to $\mathcal{R}_{\partial D}$. This extension is also denoted (with slight abuse of language) by ω_z, as in (3). A subset A of ∂D is *measurable with respect to harmonic measure for* D precisely when $\mathsf{A} \in \mathcal{R}_{\partial D}$. Hence if A is a subset of ∂D, then we may meaningfully evaluate its harmonic measure $\omega_z(\mathsf{A} \mid D)$ precisely when $\mathsf{A} \in \mathcal{R}_{\partial D}$, for only in this case is the expression $\omega_z(\mathsf{A} \mid D)$ well-defined. The σ-algebra $\mathcal{R}_{\partial D}$ is independent of z, because if A is Borel then the function $z \in D \mapsto \omega_z(\mathsf{A})$ is a (positive) harmonic function.

Example: The Unit Disc. *The harmonic measure for the unit disc* $\mathbb{U}$ *with pole at* 0 *is the normalized Lebesgue measure on its boundary (normalized so that the measure of* $\partial\mathbb{U}$ *is* 1*), and the* σ*-algebra* $\mathcal{R}_{\partial\mathbb{U}}$ *is that of Lebesgue measurable sets, generated by the Borel sets and the subsets of Borel subsets of* $\partial\mathbb{U}$ *of zero Lebesgue measure.*

Since $\mathcal{R}_{\partial D}$ is strictly larger than $\mathcal{B}(\partial D)$, we may have to evalute the harmonic measure of sets which are actually *not* Borel, and, in these cases, we first have to show that *they belong to* $\mathcal{R}_{\partial D}$.

[2]In other words, the σ-algebra $\mathcal{R}_{\partial D}$ is the measure-theoretic completion of $\mathcal{B}(\partial D)$ under harmonic measure.

The following result is very useful in dealing with these measurability issues. Recall that an analytic subset of ∂D is the continuous image of a Borel set in a Polish space. Every Borel set is analytic, but there are analytic sets that are not Borel.[3] See for example [2] or [11] for background. The relevance of analytic sets in potential theory and harmonic analysis is due to the fact that they are universally measurable, and therefore, in particular, measurable with respect to harmonic measure. The following result makes this statement more precise.

Theorem 1. *If* $\mathsf{A} \subset \partial D$ *is an analytic set, then it is measurable with respect to harmonic measure for* D*, i.e., it belongs to* $\mathcal{R}_{\partial D}$.

Proof. The result is an immediate consequence of the following theorem, due to Lusin and Sierpinski, a proof of which can be found in [6, p.751].

> **Theorem.** *Let* X *be a compact metric space, and let* μ *be a finite measure defined on the* σ*-algebra* $\mathcal{B}(X)$ *of Borel subsets of* X*. Let* $S \subset X$ *be an analytic set. Then* S *belongs to the measure-theoretic completion of* $\mathcal{B}(X)$ *with respect to* μ*, i.e., to the* σ*-algebra of sets generated by* $\mathcal{B}(X)$ *and the subsets of* μ *null sets.*

□

Accessible points

An important portion of the boundary of D is the set of points $w \in \partial D$ such that there exists a half-closed Jordan arc contained in D and ending at w. These points are called *accessible from the inside of* D. We denote by $\partial_\flat(D)$ the set of points in the boundary of D which are accessible from the inside of D. In other words, if $w \in \partial D$, then $w \in \partial_\flat(D)$ if and only if

$$w = \lim_{s \uparrow 1} x(s), \qquad x : [0,1) \to D, \tag{4}$$

where x is continuous. One may suppose that the function x is injective; see [10]. One does not assume that the path described by x is rectifiable. We say that the path x *ends at* w if (4) holds. The set $\partial_\flat$D is dense in ∂D; see [14] for $n = 2$. The questions are:

Is $\partial_\flat$D a Borel set? Does it belong to $\mathcal{R}_{\partial D}$?

We shall see that the answer to the first question depends on n, the dimension of the ambient space. Indeed, $\partial_\flat(D)$ is a natural example of a boundary set such that

[3]Lebesgue wrote that the projection on the x-axis of a Borel subset of $\mathbb{R}^2$ must *obviously* be Borel. Suslin showed that this claim is false.

- if $n > 2$, it is *not necessarily* Borel.
- it is measurable with respect to harmonic measure.

The following result shows that Theorem 1 is not only useful, but, for $n > 2$, even essential in order to show that $\partial_\flat(D) \in \mathcal{R}_{\partial D}$.

Theorem 2. *If $n > 2$, there is a bounded, connected, open set $D \subset \mathbb{R}^n$ such that $\partial_\flat$D is not Borel.*

Proof. Recall that Urysohn and Nikodym constructed a closed set $F \subset \mathbb{R}^3$ such that the set of all the points in F, which are accessible, from the outside of F, along a Jordan half-closed arc, is *not* a Borel set. This construction can be adapted to show the existence of a domain $D \subset \mathbb{R}^3$ such that $\partial_\flat$D is not Borel. See also [18]. □

Theorem 3. *The set $\partial_\flat(D)$ is an analytic set.*

Proof. Let $\mathbb{N}$ be the set of positive integers, endowed with the discrete topology. Recall that the Baire space $\mathcal{N}$ is the set $\mathbb{N}^{\mathbb{N}}$ of sequences of positive integers, endowed with the product topology. Every $\mathbf{x} \in \mathcal{N}$ is a function $\mathbf{x} : \mathbb{N} \to \mathbb{N}$, and if $k \in \mathbb{N}$, then $\mathbf{x}(k) \in \mathbb{N}$ is called the k^{th}-component of x. Subsets of $\mathbb{R}^d$ of the form

$$\left\{y \in \mathbb{R}^d : \text{ there exists } \mathbf{x} \in \mathcal{N} \text{ such that } (\mathbf{x}, y) \in R\right\},$$

where R is some Borel subset of $\mathcal{N} \times \mathbb{R}^d$, are analytic sets; they are not necessarily Borel. Fix bijective maps $f : \mathbb{N} \to \mathbb{Q} \cap [0, 1)$ and $g : \mathbb{N} \to \mathbb{Q}^d \cap D$. Observe that if $y \in \partial$D then $y \in \partial_\flat$D if and only if there is a continuous function $x : [0, 1) \to D$ ending at y and such that $x(f(n)) \in \mathbb{Q}^d$ for each $n \in \mathbb{N}$. Indeed, it suffices to choose a polygonal path ending at y consisting of straight line segments whose endpoints have rational coordinates. Let C be the set of continuous functions $x : [0, 1) \to D$ such that $x((f(n)) \in \mathbb{Q}^d$ for every $n \in \mathbb{N}$. Consider the injective function

$$C \to \mathcal{N},\, x \mapsto x^*, \qquad x^*(k) \equiv g^{-1}(x(f(k))) \qquad (k \in \mathbb{N}), \tag{5}$$

and denote by $\mathcal{C} \subset \mathcal{N}$ the set $\{x^* : x \in C\}$. Observe that

$$\mathcal{C} = \bigcap_{k\in\mathbb{N}} \bigcap_{l\in\mathbb{N}} \bigcup_{h\in\mathbb{N}} \bigcap_{p\in A_k^h} \mathcal{B}_k^l(p),$$

where

$$A_k^h \equiv \left\{p \in \mathbb{N} : |f(k) - f(p)| < \frac{1}{h}\right\},$$

and

$$\mathcal{B}_k^l(p) \equiv \left\{ \mathbf{x} \in \mathcal{N} : |g(\mathbf{x}(k)) - g(\mathbf{x}(p))| < \frac{1}{l} \right\}.$$

Since the sets $\mathcal{B}_k^l(p) \subset \mathcal{N}$ are open, $\mathcal{C}$ is Borel. If $(\mathbf{x}, y) \in \mathcal{N} \times \mathbb{R}^d$ then we say that $\mathbf{x}$ ends at y if $\mathbf{x} = x^*$ for some $x \in C$ such that x ends at y. Let

$$\mathcal{W} \equiv \bigcap_{n \in \mathbb{N}} \bigcup_{h \in \mathbb{N}} \bigcap_{p \in Z(h)} \mathcal{W}_n^p,$$

where

$$Z(h) \equiv \left\{ p \in \mathbb{N} : |f(p) - 1| < \frac{1}{h} \right\}, \mathcal{W}_n^p \equiv \left\{ (\mathbf{x}, y) \in \mathcal{N} \times \mathbb{R}^d : |g(\mathbf{x}(p)) - y| < \frac{1}{n} \right\}.$$

Then $\mathcal{W} \subset \mathcal{N} \times \mathbb{R}^d$ is Borel, since the sets $\mathcal{W}_n^p \subset \mathcal{N} \times \mathbb{R}^d$ are open. Observe that if $\mathbf{x} \in \mathcal{C}$ then $(\mathbf{x}, y) \in \mathcal{W}$ if and only if $\mathbf{x}$ ends at y. Therefore

$$\partial_\flat \mathrm{D} = \left\{ y \in \mathbb{R}^d : \text{ there exists } \mathbf{x} \in \mathcal{N} \text{ such that } (\mathbf{x}, y) \in R \right\},$$

where

$$R \equiv (\mathcal{C} \times \mathbb{R}^d) \cap (\mathcal{N} \times \partial \mathrm{D}) \cap \mathcal{W}.$$

It follows that $\partial_\flat \mathrm{D}$ is analytic, since R is a Borel subset of $\mathcal{N} \times \mathbb{R}^d$. □

The following result follows at once.

Proposition 1. *The set $\partial_\flat(D)$ is measurable with respect to harmonic measure.*

We shall see that, if $n = 2$, then more is true: $\partial_\flat \mathrm{D}$ is Borel. However, no *elementary* proof of this fact exists, as far as we know. It would be nice to find one.

However, using conformal mapping arguments, we will give a direct proof of Proposition 1 *in the case $n = 2$*. This proof is independent of the theory of analytic sets and, in particular, independent of Theorems 1 and 3.

Other notions of accessibility

The notion of accessibility employed in this work should not be confused with the one discussed by P. Urysohn [26]. If $S \subset \mathbb{R}^d$, the points $y \in S$, such that

$$y = \lim_{s \uparrow 1} x(s), \quad \text{where } x : [0,1) \to \mathbb{R}^d \setminus S \text{ is continuous,}$$

are called *accessible from* $\mathbb{R}^d \setminus S$. Denote by $\mathcal{U}(S)$ the set of all points of S which are accessible from $\mathbb{R}^d \setminus S$.

Observe that, if $D \subset \mathbb{R}^d$ is a bounded, open and connected, then

$$\partial_\flat \mathrm{D} \subset \mathcal{U}(\partial \mathrm{D}) \subset \partial \mathrm{D}$$

(where both inclusions may be strict).
It would be interesting to find an elementary proof of the following result, due to Mazurkiewicz.

Theorem (Mazurkiewicz 1936) [15]. *If $C \subset \mathbb{R}^2$ is compact, then $\mathcal{U}(C)$ is Borel.*

The proof of the following result (Proposition 2) is based on Mazurkiewicz's Theorem.

Proposition 2. *If $n = 2$ then $\partial_\flat \mathrm{D}$ is Borel.*

Proof. It suffices to observe that, if B is an open ball such that $B \supset \overline{D}$, then

$$\partial_\flat \mathrm{D} = \mathcal{U}(\overline{B} \setminus D) \setminus \partial B, \tag{6}$$

and then apply Mazurkiewicz's result. □

A direct, elementary proof of Proposition 2, independent of Mazurkiewicz's Theorem, would be of independent interest.[4]
The non-elementary nature of Mazurkiewicz's theorem can also be seen from the following result, due to Urysohn.

Theorem (Urysohn 1925). *There exist*

- *a G_δ set $Q \subset \mathbb{R}^2$ such that $\mathcal{U}(Q)$ is not an analytic set.*
- *a closed set $F \subset \mathbb{R}^3$ such that $\mathcal{U}(F)$ is not a Borel set.*

If $F \subset \mathbb{R}^n$ is closed, then $\mathcal{U}(F)$ is an analytic set.

In particular, Urysohn proved that, if $n > 2$, and $F \subset \mathbb{R}^n$ is closed, then $\mathcal{U}(F)$ is an analytic set but it is not necessarily Borel.
Observe that (6), coupled with this theorem of Urysohn, together imply Theorem 3. Our proof of Theorem 3 is new, simpler, and independent.

[4]Observe that a proof of Proposition 2 has been given in [3, p.481] using a more difficult argument, based on conformal mappings. See also [4].

The accessibile-points operator

All these notions may be subsumed under just one operator. A point $w \in \mathbb{R}^n$ *is accessible from* $S \subset \mathbb{R}^n$ if $w = \lim_{s\uparrow 1} c(s)$ for a continuous and injective function $c : [0,1) \to S$. The set of all points in $\mathbb{R}^n$ which are accessible from S is denoted by $\mathcal{A}(S)$. Then

$$\mathcal{U}(S) = S \cap \mathcal{A}(\mathbb{R}^n \setminus S)$$

and

$$\partial_\flat \mathrm{D} = \partial \mathrm{D} \cap \mathcal{A}(D).$$

Rectilinear accessibility

The notion of accessibility employed in this work should not be confused with that of *rectilinear accessibility*, introduced in [26] and also discussed for example in [2] and in [18]. See also [24]. Recall that a point $y \in \partial \mathrm{D}$ is said to be *rectilinearly accessible from* $\mathbb{R}^d \setminus \partial \mathrm{D}$ if

$$y = \lim_{s\uparrow 1} x(s), \quad \text{where } x : [0,1) \to \mathbb{R}^d \setminus \partial \mathrm{D} \text{ and } x(s) \equiv z + s\,w \text{ for some } w, z \in \mathbb{R}^d.$$

This notion appears to have little interest from the point of view of potential theory. Urysohn proves that for a closed set the set of rectilinearly accessible points is an analytic set as well. One can easily prove this fact by methods similar to those of the proof of theorem 3.

The Riemann mapping

Theorem (Riemann 1851) [22]. *If $D \subset \mathbb{R}^2$ is bounded, open, connected, and simply connected, then there exists a surjective and conformal mapping*

$$\varphi : \mathbb{U} \to D,$$

where $\mathbb{U}$ is the unit disc $\mathbb{U} \overset{\text{def}}{=} \{z \in \mathbb{C} : |z| < 1\}$.

The function φ is called a *Riemann mapping* of D. A Riemann mapping of D induces a correspondence between $\partial\mathbb{U}$ and $\partial\mathrm{D}$: More precisely, the function

$$\varphi_\flat : \mathcal{U} \to \partial \mathrm{D},$$

(where $\mathcal{U} \subset \partial\mathbb{U}$ is a certain Borel subset of full Lebesgue measure) is defined in terms of the angular limiting values of φ. More precisely, the set $\mathcal{U}$ is defined in terms of the following notion, which is basic for the boundary behaviour of functions $f : \mathbb{U} \to \mathbb{C}$. A point $\zeta \in \partial\mathbb{U}$ is a *Fatou point of* $f : \mathbb{U} \to \mathbb{C}$ if f has an *angular limit at* ζ, i.e., if $f(z)$ converges to some limit, which may be ∞, and which is denoted by $f_\flat(\zeta)$, as $z \in \mathbb{U}$ converges to ζ inside triangles, contained in $\mathbb{U}$, having vertex at ζ; see [5]. The set of all Fatou points of f is denoted by $\mathcal{U}(f)$.

Lemma ([20], p.131). *If* $\mathbb{U} \xrightarrow{f} \mathbb{C}$ *is continuous then*

$$\mathcal{U}(f) \text{ is a Borel subset of } \partial\mathbb{U}. \tag{7}$$

Fatou's Theorem ([7]). *If* $\varphi : \mathbb{U} \to \mathbb{C}$ *is a bounded analytic function, then the set* $\partial\mathbb{U} \setminus \mathcal{U}(\varphi)$ *has zero Lebesgue measure.*

We write $\mathcal{U} = \mathcal{U}(\varphi)$

Direct images. If φ is a Riemann mapping of $D \subset \mathbb{R}^2$, and $\mathsf{U} \subset \partial\mathbb{U}$, then the *direct image of* U *under* φ is the subset of $\partial\mathrm{D}$ given by

$$\varphi_\star[\mathsf{U}] \stackrel{\text{def}}{=} \{\varphi_\flat(\zeta) : \zeta \in \mathsf{U} \cap \mathcal{U}\} \subset \partial\mathrm{D}.$$

See [16].

Observe that the function $\varphi_\flat$ is not, in general, surjective. The direct image of $\partial\mathbb{U}$ is independent of φ, and it has an intrinsic meaning.

Lemma ([20] Theorem 4.3, [14] Theorems III.2.6 and III.2.7.). *If* φ *is a Riemann mapping of* $D \subset \mathbb{R}^2$, *then*

$$\varphi_\star[\partial\mathbb{U}] = \varphi_\star[\mathcal{U}] = \partial_\flat\mathrm{D}. \tag{8}$$

Inverse images. If φ is a Riemann mapping of $D \subset \mathbb{R}^2$, and $\mathsf{A} \subset \partial\mathrm{D}$, then the *inverse image of* A *under* φ is the subset of $\partial\mathbb{U}$ given by

$$\varphi^\star[\mathsf{A}] \stackrel{\text{def}}{=} \{w \in \mathcal{U} : \varphi_\flat(w) \in \mathsf{A}\} \subset \partial\mathbb{U}.$$

We are interested in the following equality

$$\omega_{\varphi(0)}(\mathsf{A} \mid D) = \omega_0(\varphi^\star[\mathsf{A}] \mid \mathbb{U}). \tag{9}$$

The question is:

For which subsets A of ∂D does (9) hold?

The left-hand side of (9) makes sense precisely when

$$\mathsf{A} \text{ is measurable with respect to harmonic measure for } D \qquad (\mathsf{A} \in \mathcal{R}_{\partial D})$$

while the right-hand side makes sense precisely when

$$\varphi^\star[\mathsf{A}] \text{ is Lebesgue measurable.} \qquad (\varphi^\star[\mathsf{A}] \in \mathcal{R}_{\partial\mathbb{U}})$$

The question is: Do we need to assume that both conditions hold, in order to be able to deduce (9)? Are the conditions $(\mathsf{A} \in \mathcal{R}_{\partial D})$ and $(\varphi^\star[\mathsf{A}] \in \mathcal{R}_{\partial\mathbb{U}})$ equivalent? Let us first review the known results.

Theorem A. *Let* $\mathsf{A} \subset \partial\mathrm{D}$. *Then*

$$\mathsf{A} \subset \partial\mathrm{D} \textit{ is Borel} \Longrightarrow \varphi^\star[\mathsf{A}] \textit{ is a Borel subset of } \partial\mathbb{U}, \textit{ and } (9) \textit{ holds.}$$

A proof that

$$\mathsf{A} \subset \partial\mathrm{D} \text{ is Borel} \Longrightarrow \varphi^\star[\mathsf{A}] \text{ is a Borel subset of } \partial\mathbb{U} \tag{10}$$

can be found in [20, p.131]. This means that, if $\mathsf{A} \subset \partial\mathrm{D}$ is Borel, then both sides of (9) are well-defined. A proof that (9) holds *under the assumption that* A *is Borel,* can be found in [9, p.207]. Under the assumption that ∂D is smooth enough, this fact follows from the conformal invariance of harmonic functions. The proof for arbitrary domains, without restrictions on their boundary smoothness, is handled by a limiting argument.
An immediate consequence of Theorem A is the following:

Theorem B. *Let* $\mathsf{A} \subset \partial D$. *Then*

$$(\mathsf{A} \in \mathcal{R}_{\partial D}) \Longrightarrow (\varphi^\star[\mathsf{A}] \in \mathcal{R}_{\partial\mathbb{U}}), \textit{ and } (9) \textit{ holds.}$$

Theorem B follows at once from Theorem A using the fact that $\mathcal{R}_{\partial D}$ is the measure-theoretic completion of $\mathcal{B}(\partial D)$ under harmonic measure. The details are left to the reader.
In order to complete the picture, it suffices to prove the following statement.[5]

Theorem 4. *Let* $\mathsf{A} \subset \partial\mathrm{D}$. *Then*

$$(\varphi^\star[\mathsf{A}] \in \mathcal{R}_{\partial\mathbb{U}}) \Longrightarrow (\mathsf{A} \in \mathcal{R}_{\partial D}).$$

Proof. The proof is based on the following:

Lemma 1. *Let* $\mathsf{A} \subset \partial D$. *If* $\varphi^\star[\mathsf{A}]$ *is Lebesgue measurable, and* $\omega_0(\varphi^\star[\mathsf{A}] \mid \mathbb{U}) < 1$, *then for every sufficiently small* $\epsilon > 0$ *there exists an open set* $G_\epsilon \subset \partial D$ *such that* $\mathsf{A} \subset G_\epsilon$ *and*

$$\omega_{\varphi(0)}(G_\epsilon \mid D) < \epsilon + \omega_0(\varphi^\star[\mathsf{A}] \mid \mathbb{U}).$$

[5]The result was stated, without proof, in [1]. As far as we know, the result has neither been stated, nor proved, anywhere else. In an initial attempts to derive Theorem 4 from Theorem B using purely measure-theoretic tools, we encountered Proposition 14 in [23, p. 341], which, if true, could have been helpful. However, an example, due to Tim Steger [25], presented in Appendix B, shows that Proposition 14 in [23, p. 341] is false.

Proof. Let $\mathsf{B} = \varphi^\star[\mathsf{A}]$. Let $\epsilon > 0$ be small enough. A Theorem of Lusin [23, p.74] theorem implies that there exists a closed set $\mathsf{F} \subset \partial\mathbb{U} \setminus \mathsf{B}$ such that

- the restriction of $\varphi_\flat$ to F is continuous;
- $\omega_0(\partial\mathbb{U} \setminus \mathsf{B} \mid \mathbb{U}) - \epsilon < \omega_0(\mathsf{F} \mid \mathbb{U})$.

It follows that $\varphi_\star[\mathsf{F}] \subset \partial D$ is closed, and, by Theorem A, that $\varphi^\star[\varphi_\star[\mathsf{F}]]$ is Lebesgue measurable and that

$$\omega_0(\varphi^\star[\varphi_\star[\mathsf{F}]] \mid \mathbb{U}) = \omega_{\varphi(0)}(\varphi_\star[\mathsf{F}] \mid D).$$

Then the set $G_\epsilon \overset{\text{def}}{=} \partial D \setminus \varphi_\star[\mathsf{F}]$ has the required properties. □

An immediate corollary of Lemma 1 is the following:

Lemma 2. *Let $\mathsf{A} \subset \partial D$. Assume that $\varphi^\star[\mathsf{A}]$ is Lebesgue measurable, and that $0 < \omega_0(\varphi^\star[\mathsf{A}] \mid \mathbb{U}) < 1$. Then for every sufficiently small $\epsilon > 0$ there exist a closed set $\mathsf{F}_\epsilon \subset \mathsf{A}$ and an open set $\mathsf{G}_\epsilon \supset \mathsf{A}$ such that*

$$\omega_{\varphi(0)}(\mathsf{G}_\epsilon \setminus \mathsf{F}_\epsilon \mid D) < \epsilon.$$

Now we are ready to prove Theorem 4.
First assume that

$$0 < \omega_0(\varphi^\star[\mathsf{A}] \mid \mathbb{U}) < 1.$$

Then an application of Lemma 2 yields a sequence of closed sets $\mathsf{F}_n \subset \mathsf{A}$ and a sequence of open sets $\mathsf{G}_n \supset \mathsf{A}$ such that $\omega_{\varphi(0)}(\mathsf{G}_n \setminus \mathsf{F}_n \mid D) < 1/n$. Then

$$\mathsf{A} = (\cup_n \mathsf{F}_n) \cup (\mathsf{A} \setminus \cup_n \mathsf{F}_n)$$

where $\cup_n \mathsf{F}_n$ is Borel, and $\mathsf{A} \setminus \cup_n \mathsf{F}_n$ is contained in

$$\cap_k (\mathsf{G}_k \setminus \mathsf{F}_k).$$

The latter set is a Borel set of harmonic measure zero. Hence A is measurable with respect to harmonic measure for D.
Now assume that

$$0 = \omega_0(\varphi^\star[\mathsf{A}] \mid \mathbb{U}).$$

An application of Lemma 1 yields the existence of a sequence of open sets $\mathsf{G}_n \subset \partial D$ containing A such that $\omega_{\varphi(0)}(\mathsf{G}_n \mid D) < 1/n$. Therefore A is contained in $\cap_n \mathsf{G}_n$, and the latter is a Borel set and it has harmonic measure equal to zero. Hence A is measurable with respect to harmonic measure for D.
The case where $\omega_0(\varphi^\star[\mathsf{A}] \mid \mathbb{U}) = 1$ is treated by symmetry, since $\partial\mathbb{U} \setminus \varphi^\star[\mathsf{A}]$ is equal to $\varphi^\star[\partial D \setminus \mathsf{A}]$. □

Observe that an immediate corollary of Theorem 4 is the following result, whose interest lies in the fact that the proof we are able to give is independent of Theorems 1 and 3, and also independent of Mazurkiewicz's theorem.

Corollary 1. *If $n = 2$ and D is simply connected, then $\partial_b \mathrm{D}$ is measurable with respect to harmonic measure for D.*

Proof. It suffices to apply Theorem 4 to the case $\mathsf{A} = \partial_b \mathrm{D}$, and observe that $\varphi^\star[\partial_b \mathrm{D}] = \mathcal{U}$. □

Appendix A

In this appendix we give a more precise definition of harmonic measure. The original proof of the Riemann mapping theorem was based on potential-theoretic tools, and reduced matter to the *classical Dirichlet problem for D.*

Regular functions for the Dirichlet problem. *We say that $f \in \mathrm{C}(\partial \mathrm{D})$ is regular on $\partial \mathrm{D}$, for the Dirichlet problem on D, if there is a (necessarily unique) function $u \in \mathrm{h}(D)$ such that*

$$\lim_{D \ni z \to w} u(z) = f(w) \ \text{ for all } w \in \partial \mathrm{D}\,. \tag{11}$$

The set of all functions in $\mathrm{C}(\partial \mathrm{D})$ regular on $\partial \mathrm{D}$ is denoted by $\mathrm{C}_{\mathrm{h}}(\partial \mathrm{D})$.

The maximum principle for harmonic functions implies that the *classical solution operator*

$$\mathrm{C}_{\mathrm{h}}(\partial \mathrm{D}) \xrightarrow{\mathrm{P}_D} \mathrm{h}(D), \quad f \mapsto \mathrm{P}_D[f] \stackrel{\text{def}}{=} u$$

is positive and linear, where $\mathrm{P}_D[f]$ is the (unique) function u appearing in the definition of regularity. The set $\mathrm{C}_{\mathrm{h}}(\partial \mathrm{D})$ is a closed subspace of $\mathrm{C}(\partial \mathrm{D})$.

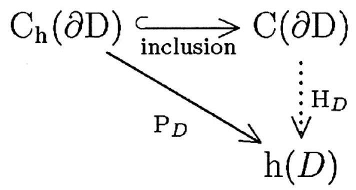
$$\begin{array}{ccc} \mathrm{C}_{\mathrm{h}}(\partial \mathrm{D}) & \xrightarrow[\text{inclusion}]{} & \mathrm{C}(\partial \mathrm{D}) \\ & \searrow^{\mathrm{P}_D} & \vdots\,? \\ & & \mathrm{h}(D) \end{array}$$

For some domains, not every boundary continuous function is regular. In these cases, it is *not* possible to extend P_D to $\mathrm{C}(\partial \mathrm{D})$ *and preserve* (11), *at the same time.*

Theorem (Zaremba 1911, Lebesgue 1912). *There exist domains D for which $\mathrm{C}_{\mathrm{h}}(\partial \mathrm{D})$ is a proper subspace of $\mathrm{C}(\partial \mathrm{D})$.*

See also [13]. The following result is remarkable. Indeed, it is plausible that an extension of P_D to $\mathrm{C}(D)$ should exist, but no special meaning could *a priori* be attached to it, if it were not unique: The uniqueness shows that the extension has intrinsic meaning.[6]

Theorem (Keldysh 1945, Perron 1923, Remak 1924, Wiener 1924). *The classical solution operator* P_D *has one and only one positive and linear extension to* $\mathrm{C}(\partial \mathrm{D})$.



The first constructions of the extension appear in [19], [21], [27], based on various constructive methods. The uniqueness is due to Keldysh [12], a work which, remarkably enough, is mostly ignored by the main treatises on potential theory (see e.g. [6] and [9]). See [17] for a self-contained treatment.

Theorem (Kellogg 1928, Evans 1933). *Let us denote by* H_D *be the unique positive and linear extension of* P_D *to* $\mathrm{C}(\partial \mathrm{D})$. *Then the points* $w \in \partial \mathrm{D}$ *for which* (11) *fails, with* $u = \mathrm{H}_D[f]$, *form a set of capacity zero.*

Let H_D be the unique positive and linear extension of P_D to $\mathrm{C}(\partial \mathrm{D})$. For each $z \in D$, there is a unique[7] Borel measure ω_z on $\partial \mathrm{D}$ such that

$$\mathrm{H}_D(f)(z) = \int_{\partial \mathrm{D}} f(w)\, \omega_z(dw \mid D), \quad \forall f \in \mathrm{C}(\partial \mathrm{D}). \tag{12}$$

The Borel measure ω_z is (by definition) the harmonic measure for D with pole at z.[8] See [27].

Let us denote by $\mathcal{D}_n$ the set of all bounded, open, connected subsets of $\mathbb{R}^n$. The set $\mathcal{D}_n$ is partially ordered by the relation $\prec$ defined as follows: $D_1 \prec D_2$ if and only if $\overline{D_1} \subset D_2$ (where $\overline{D}$ denotes the closure of D). The set $\mathcal{D}_n$ has a natural topology (induced by the partial order $\prec$), that is useful when we look at

[6] Some authors introduce harmonic measure using the Hahn-Banach theorem, a method which clearly does not yield uniqueness.

[7] See [8, Theorem (7.2) p. 205].

[8] As observed in [8, p.208], the proof of the Riesz Representation Theorem yields a σ-algebra of measurable sets, which turns out to be precisely the completion of the Borel σ-algebra. The two approaches are perfectly equivalent.

harmonic measure *as a function of the domain.* Indeed, the proof of Theorem A rests on an approximation result where one makes an *implicit* use of this topology. A basis of open sets in $\mathcal{D}_n$ is given by the intervals

$$(D_1; D_2) \stackrel{\text{def}}{=} \{D \in \mathcal{D}_n : D_1 \prec D \prec D_2\}$$

for any D_1 and D_2 in $\mathcal{D}_n$ such that $D_1 \prec D_2$.

Appendix B

The following example, due to Tim Steger [25], shows that Proposition 14 [23, p.341] is false.
Let $A \subset [0,1]$ a set of Lebesgue outer measure equal to 1, such that $[0,1] \setminus A$ has Lebesgue outer measure equal to 1. Let $\nu^*(E)$ be defined as the Lebesgue outer measure of the set $E \cap A$. Then ν^* is an outer measure on $[0,1]$, in the sense of [23]. Let $\mathcal{M}$ be the σ-algebra of all ν^*-measurable subsets of $[0,1]$, and let ν be the restriction of ν^* to $\mathcal{M}$. Then $\mathcal{M}$ contains all the Borel subsets of $[0,1]$. The measure space $([0,1], \mathcal{M}, \nu)$ satisfies the hypothesis in Proposition 14 [23, p.341] but not its conclusion.

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